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Normal indices in Nikishin systems ☆

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Abstract

We improve the class of indices for which normality takes place in a Nikishin system and apply this in Hermite–Padé approximation of such systems of functions.

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1. Introduction

Let $\mathbf{f} = (f_1, \ldots, f_m)$ be a finite system of (formal) power series

$$f_j(z) = \sum_{k=0}^{\infty} \frac{c_{j,k}}{z^{k+1}}, \quad j = 1, \ldots, m.$$

Fix a multi-index $n = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ where $\mathbb{Z}_+ = \{0, 1, \ldots\}$, and set $|n| = n_1 +$

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$n_2 + \cdots + n_m$. It is easy to see that there exists a polynomial Q_n such that

$$\text{(i)}\quad Q_n(z) \not\equiv 0, \quad \deg Q_n \leqslant |n|,$$

$$\text{(ii)}\quad (Q_n f_j - P_{n,j})(z) = \frac{A_{n,j}}{z^{n_j+1}} + \cdots, \quad j = 1, \ldots, m, \tag{1}$$

where on the right-hand side of (ii) we have a formal series in increasing powers of $1/z$ and $P_{n,j}$ is the polynomial part of the power expansion of $Q_n f_j$ at $z = \infty$ (hence $\deg P_{n,j} \leqslant |n| - 1$). The construction of Q_n, reduces to finding a non-trivial solution of a homogeneous linear system of $|n|$ equations on $|n| + 1$ unknowns (the coefficients of Q_n). Therefore, a non-trivial solution always exists. For each solution of (1), the vector $\left(\frac{P_{n,1}}{Q_n}, \ldots, \frac{P_{n,m}}{Q_n}\right)$ is called the Hermite–Padé approximant (or simultaneous Padé approximant) of $(f_1, \ldots, f_m)$ relative to the multi-index $(n_1, \ldots, n_m)$. In the case of one function, the definition reduces to that of a diagonal Padé approximant.

It is well known that Padé approximants and, in particular, diagonal Padé approximants are uniquely determined. This is not the case for Hermite–Padé approximants when $m \geqslant 2$. Different solutions to the homogeneous system mentioned above can give rise to different vector Hermite–Padé approximants.

From (1) it follows that a sufficient condition in order that the multi–index $(n_1, \ldots, n_m)$ determines a unique vector Padé approximant, is to be able to ensure that any Q_n which solves (1) has $\deg Q_n = |n|$. In fact, if Q_n and $\tilde{Q}_n$ satisfy (ii), we would have that $\tilde{Q}_n = \lambda Q_n$, $\lambda \neq 0$, since otherwise we can obtain a polynomial of degree less than $|n|$ that verifies (i) and (ii). Multi-indices n for which any solution of (1) has $\deg Q_n = |n|$ are said to be normal. A system of functions $(f_1, \ldots, f_m)$ is said to be perfect if all multi-indices are normal. Normality of indices and perfectness of systems are key problems in number theory applications of Hermite–Padé approximants.

There are two types of systems for which the theory is fairly well developed. The so-called Angelesco systems [1] which are known to be perfect and the Nikishin systems [7]. They will be defined in the next section. The main result of this paper (see Theorem 1) consists in proving that in a Nikishin system all multi-indices $n = (n_1, \ldots, n_m)$ for which there do not exist $i < j < k$ such that $n_i < n_j < n_k$ are normal. In particular, a Nikishin system of two functions is perfect and one with three functions has all indices normal except (possibly) when $n_1 < n_2 < n_3$. The question of whether Nikishin systems are perfect or not remains open. The concept of Nikishin system was extended in a recent paper by Gonchar et al. [5]. Such systems combine Nikishin and Angelesco systems. Our technique can also be applied in that setting to improve the class of indices known for which normality takes place in the generalized case.

2. Angelesco and Nikishin systems

In the sequel, we study Hermite–Padé approximants for systems of Markov-type functions. Let

$$f_j(z) = \hat{s}_j(z) = \int \frac{ds_j(x)}{z-x}, \quad j = 1, \ldots, m, \tag{2}$$

where s_j are finite Borel measures supported on the real line with constant sign. We do not require that the supports be compact sets, but we will assume that for all $j = 1, \ldots, m$ and $k \in \mathbb{Z}_+$

$$\int |x|^k \, ds_j(x) < +\infty.$$

It is easy to verify that solving (1), for the system $(f_1, \ldots, f_m)$ given by (2), is equivalent to finding $Q_n \not\equiv 0, \deg Q_n \leqslant |n|$ such that

$$\int x^k Q_n(x) \, ds_j(x) = 0, \quad k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, m. \tag{3}$$

If the supports of the measures are compact sets, (3) follows immediately by use of Cauchy's integral formula and (ii). In the case of unbounded support (or for what matters in any case) the proof of the corresponding relations in (3) is a direct consequence of the algebraic interpretation of (ii) (cf. [2, Lemma 3]).

Let $\mathrm{supp}(s_j)$ denote the support of s_j and $\Delta_j = \mathscr{C}_0(\mathrm{supp}(s_j))$ be the smallest interval (bounded or unbounded) containing $\mathrm{supp}(s_j)$. When $\Delta_j \cap \Delta_k = \emptyset, j \neq k$, we are in the presence of an Angelesco system. From (3) we have that Q_n has at least n_j distinct zeros on Δ_j. If we have an Angelesco system, it follows immediately that $\deg Q_n = |n|$ and all the zeros of Q_n are simple since the zeros attributed lie on different intervals. Thus, the perfectness of Angelesco systems is quite trivial. When the intervals overlap the problem becomes quite difficult. In a Nikishin system, the supports of all the measures s_j coincide and, therefore, so do the intervals Δ_j.

Before defining a Nikishin system, we need some additional notation. We adopt the one introduced in [5] which is very simple and clarifying. Let σ_1 and σ_2 be measures on $\mathbb{R}$ and let F_1 and F_2 be the smallest intervals containing $\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\sigma_2)$, respectively. Assume that $F_1 \cap F_2 = \emptyset$. We define the measure $\langle \sigma_1, \sigma_2 \rangle$ as follows:

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t} \, d\sigma_1(x) = \hat{\sigma}_2(x) \, d\sigma_1(x). \tag{4}$$

Therefore, $\langle \sigma_1, \sigma_2 \rangle$ is a measure with constant sign and support equal to that of σ_1.

For a system of closed intervals $F_1, F_2, \ldots, F_m$ such that $F_{j-1} \cap F_j = \emptyset, j = 2, \ldots, m$ and measures $\sigma_1, \sigma_2, \ldots, \sigma_m$ with $\mathscr{C}_0(\mathrm{supp}(\sigma_j)) = F_j$, we define by induction the measures

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{j+1} \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_{j+1} \rangle \rangle, \quad j = 2, \ldots, m-1.$$

We assume that for all $k \in \mathbb{Z}_+$ and $1 \leqslant j \leqslant m$, $|x|^k \in L_1(\sigma_j)$. Set

$$s_1 = \langle \sigma_1 \rangle, \; s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, s_m = \langle \sigma_1, \ldots, \sigma_m \rangle.$$

We say that $(\hat{s}_1, \ldots, \hat{s}_m)$ is the Nikishin system associated with $(\sigma_1, \ldots, \sigma_m)$. Notice that the ordering of the measures is important.

The following lemma was proved in [3, Theorem 4.1] and is a particular case of [5, Proposition 3]. In these papers, the measures $\sigma_j, 1 \leqslant j \leqslant m$, have compact support but except for minor details the same proofs follow through assuming that $|x|^k \in L_1(\sigma_j)$.

Lemma 1. *Let* $\mathbf{f} = (\hat{s}_1, \dots, \hat{s}_m)$ *be a Nikishin system and* $n = (n_1, \dots, n_m)$ *a multi-index such that*

$$1 \leqslant i < j \leqslant m \Rightarrow n_j \leqslant n_i + 1. \tag{5}$$

Then, Q_n *has exactly* $|n| = n_1 + \cdots + n_m$ *simple zeros in* F_1. *Therefore,* $\deg Q_n = |n|$ *and the multi-index* n *is normal.*

3. Main results

Definition 1. Let $\mathbf{f}$ and $\tilde{\mathbf{f}}$ be two Nikishin systems and let n and $\tilde{n}$ be two multi-indices. We say that the pairs $(\mathbf{f}, n)$ and $(\tilde{\mathbf{f}}, \tilde{n})$ are equivalent, if the common denominator $\tilde{Q}$ of the Hermite–Padé approximant of $\tilde{\mathbf{f}}$ relative to $\tilde{n}$, satisfies the same collection of orthogonality relations (3), as the common denominator Q of the Hermite–Padé approximant of $\mathbf{f}$ relative to n, and vice versa.

We introduce some simple transformations which allow to transform pairs $(\mathbf{f}, n)$ into equivalent ones. Since the definition above obviously determines an equivalence relation, if after a finite number of such transformations we arrive to a pair $(\tilde{\mathbf{f}}, \tilde{n})$ which satisfies condition (5) of Lemma 1, we can assure that the index n is normal for $\mathbf{f}$.

Let σ_α be a finite measure supported on the real line. It is well known (see [6, Appendix]) that there exists a finite measure τ_α such that $\mathscr{C}_0(\operatorname{supp}(\tau_\alpha)) \subset \mathscr{C}_0(\operatorname{supp}(\sigma_\alpha))$ and

$$\frac{1}{\hat{\sigma}_\alpha(z)} = l_\alpha(z) + \hat{\tau}_\alpha(z), \quad z \in \mathbb{C} \setminus \operatorname{supp} \sigma_\alpha, \tag{6}$$

where $l_\alpha(z) = a_\alpha z + b_\alpha$ is a polynomial of degree 1.

Given a measurable function g on $\operatorname{supp} \sigma_\alpha$, by $g\sigma_\alpha$ we denote the measure given by $g(x)\ d\sigma_\alpha(x)$.

Lemma 2. *Let* $(\sigma_1, \dots, \sigma_m)$ *be a system of measures defining a Nikishin system. Let* $2 \leqslant j \leqslant m$, *then*

$$\langle \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j \rangle = \sigma_{j-1} - l_j \langle \sigma_{j-1}, \sigma_j \rangle. \tag{7}$$

Suppose that $j < m$, *then*

$$\langle \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j, g \langle \sigma_{j+1}, \sigma_j \rangle \rangle = -\langle \sigma_{j-1}, \sigma_j, g\sigma_{j+1} \rangle + A_g \langle \sigma_{j-1}, \sigma_j \rangle \tag{8}$$

for any $g \in L_1(\sigma_{j+1})$, *where* $A_g = -a_j \int g(t) \hat{\sigma}_j(t) d\sigma_{j+1}(t)$.

Proof. Let us start out proving (7). From (4) and (6), the left-hand side of (7) is

$$\begin{aligned} & d\langle \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j \rangle(x) \\ &= \hat{\tau}_j(x)\hat{\sigma}_j(x)\, d\sigma_{j-1}(x) = \left[\frac{1}{\hat{\sigma}_j(x)} - l_j(x)\right]\hat{\sigma}_j(x)\, d\sigma_{j-1}(x) \\ &= d\sigma_{j-1}(x) - l_j(x)\hat{\sigma}_j(x)\, d\sigma_{j-1}(x) = d\sigma_{j-1}(x) - l_j(x)\, d\langle \sigma_{j-1}, \sigma_j \rangle(x). \end{aligned}$$

The last expression in this chain of equalities is the differential form of the measure in the right-hand side of (7).

In order to prove (8), first let us obtain a convenient expression for the differential form of the measure $\langle \tau_j, g\langle \sigma_{j+1}, \sigma_j \rangle \rangle$. Let y denote the variable on $\operatorname{supp} \sigma_j$ and t the variable on $\operatorname{supp} \sigma_{j+1}$. By (4) we have that

$$d\langle \tau_j, g\langle \sigma_{j+1}, \sigma_j \rangle \rangle(y) = \int \frac{g(t)\hat{\sigma}_j(t)\, d\sigma_{j+1}(t)}{y-t}\, d\tau_j(y).$$

Let x be the variable on $\operatorname{supp} \sigma_{j-1}$. From the previous relation and (4) we have that

$$\begin{aligned} & d\langle \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j, g\langle \sigma_{j+1}, \sigma_j \rangle \rangle(x) \\ &= d\langle \langle \sigma_{j-1}, \sigma_j \rangle, \langle \tau_j, g\langle \sigma_{j+1}, \sigma_j \rangle \rangle \rangle(x) \\ &= \int \frac{1}{x-y} \int g(t) \frac{\hat{\sigma}_j(t)\, d\sigma_{j+1}(t)}{y-t}\, d\tau_j(y)\hat{\sigma}_j(x)\, d\sigma_{j-1}(x) \\ &= \int \int \frac{g(t)\hat{\sigma}_j(t)\, d\sigma_{j+1}(t) d\tau_j(y)}{(x-y)(y-t)}\, \hat{\sigma}_j(x)\, d\sigma_{j-1}(x). \end{aligned}$$

Let us transform the last expression using the identity

$$\frac{1}{(x-y)(y-t)} = \frac{1}{t-x}\left[\frac{1}{t-y} - \frac{1}{x-y}\right]. \tag{9}$$

Applying Fubini's theorem and (6), we get

$$\begin{aligned} & d\langle \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j, g\langle \sigma_{j+1}, \sigma_j \rangle \rangle(x) \\ &= \int \frac{g(t)\hat{\sigma}_j(t)}{t-x} \int \left[\frac{1}{t-y} - \frac{1}{x-y}\right] d\tau_j(y)\, d\sigma_{j+1}(t)\hat{\sigma}_j(x)\, d\sigma_{j-1}(x) \\ &= \int \frac{g(t)\hat{\sigma}_j(t)\hat{\sigma}_j(x)}{t-x}\left[\frac{1}{\hat{\sigma}_j(t)} - \frac{1}{\hat{\sigma}_j(x)} + l_j(x) - l_j(t)\right] d\sigma_{j+1}(t)\, d\sigma_{j-1}(x) \\ &= \int \left[\frac{g(t)(\hat{\sigma}_j(x) - \hat{\sigma}_j(t))}{t-x} - a_j f(t)\hat{\sigma}_j(t)\hat{\sigma}_j(x)\right] d\sigma_{j+1}(t)\, d\sigma_{j-1}(x), \end{aligned}$$

where $l_j(z) = a_j z + b_j$ is the first degree polynomial which appears in (6). Then, it follows that the left-hand side of (8) is equal to

$$\int \frac{g(t)\big(\hat{\sigma}_j(x) - \hat{\sigma}_j(t)\big)}{t-x}\, d\sigma_{j+1}(t)\, d\sigma_{\alpha_-}(x) + A_g\hat{\sigma}_j(x)\, d\sigma_{j-1}(x)$$

with A_g as indicated in (8). Thus, the second term in the last expression corresponds to the second term on the right-hand side of (8). To complete the proof it is sufficient to show that the first term in the last expression equals the first term on the right-hand side of (8). To this end, using again (9) in the opposite direction and Fubini's theorem, we have

$$\int \frac{g(t)(\hat{\sigma}_j(x)-\hat{\sigma}_j(t))}{t-x}\,d\sigma_{j+1}(t)\,d\sigma_{j-1}(x)$$
$$= \int \frac{g(t)}{t-x}\left[\int \frac{d\sigma_j(y)}{x-y} - \int \frac{d\sigma_j(y)}{t-y}\right] d\sigma_{j+1}(t)\,d\sigma_{j-1}(x)$$
$$= \int\int \frac{g(t)}{t-x}\left[\frac{1}{x-y} - \frac{1}{t-y}\right] d\sigma_j(y)\,d\sigma_{j+1}(t)\,d\sigma_{j-1}(x)$$
$$= -\int\int \frac{g(t)d\sigma_{j+1}(t)}{(x-y)(y-t)}\,d\sigma_j(y)\,d\sigma_{j-1}(x),$$

which is the differential expression of the measure $-\langle \sigma_{j-1}, \sigma_j, g\sigma_{j+1} \rangle$ as we needed to prove. □

Lemma 3. *Let σ_α and σ_β be two measures and $l(z) = az + b$ a first degree polynomial. We have*

$$\langle \sigma_\alpha, l\sigma_\beta \rangle = A\langle \sigma_\alpha \rangle + \langle l\sigma_\alpha, \sigma_\beta \rangle,$$

where $A = -a|\sigma_\beta|$ and $|\sigma_\beta| = \int d\sigma_\beta(t)$ denotes the total variation of σ_β.

Proof. In fact,

$$d\langle \sigma_\alpha, l\sigma_\beta \rangle(x) = \int \frac{l(t)\,d\sigma_\beta(t)}{x-t}\,d\sigma_\alpha(x)$$
$$= \int \frac{(l(t)-l(x))\,d\sigma_\beta(t)}{x-t}\,d\sigma_\alpha(x) + \int \frac{l(x)\,d\sigma_\beta(t)}{x-t}\,d\sigma_\alpha(x)$$
$$= -a\int d\sigma_\beta(t)\,d\sigma_\alpha(x) + d\langle l\sigma_\alpha, \sigma_\beta \rangle(x)$$

as claimed. □

Let **f** be a Nikishin system and let n be a multi-index. Given a fixed $j, 2 \leqslant j \leqslant m$, assume that n satisfies

(a) $n_{j-1} < n_j$ and $n_i \geqslant n_{j-1}, 1 \leqslant i \leqslant j-1$,
(b) if $j < m$ then $n_i \leqslant n_j, j+1 \leqslant i \leqslant m$.

Let us define the following transformation. Let $(\mathbf{f}, n)$ satisfy (a) and (b). Let $(\sigma_1, \ldots, \sigma_m)$ be the collection of measures defining **f**. By $\mathbf{f}^j$ we denote the Nikishin system generated by the collection of measures which is obtained from the previous one assigning to the coordinate $j-1$ the measure $\langle \sigma_{j-1}, \sigma_j \rangle$ to the coordinate j the measure τ_j and if $j < m$ to the coordinate $j+1$ we assign the measure $\langle \sigma_{j+1}, \sigma_j \rangle$. The other measures remain unchanged. By n^j we denote the multi-index obtained from n interchanging the coordinates $j-1$ and j.

Lemma 4. *Let* $(\mathbf{f}, n)$ *be given and let n satisfy* (a) *and* (b) *for a given* $j, 2 \leqslant j \leqslant m$. *Then* $(\mathbf{f}^j, n^j)$ *is equivalent to* $(\mathbf{f}, n)$.

Proof. Let us check that both systems verify the same orthogonality relations (3). For the second system, we denote $n^j = (\tilde{n}_1, \dots, \tilde{n}_m)$ whereas $\mathbf{f}^j$ is given by the Cauchy transform of the measures $\tilde{s}_1, \dots, \tilde{s}_m$. For $1 \leqslant i < j-1$ we have that $\tilde{s}_i = s_i$ and $\tilde{n}_i = n_i$ so trivially for such indices i the same orthogonality relations hold. Also, $\tilde{s}_{j-1} = s_j$ and $\tilde{n}_{j-1} = n_j$ and thus the index $j-1$ in the second system gives the same orthogonality relations as the index j for the first system.

Now, let us consider the index j. On the second system, this index is associated to the measure

$$\tilde{s}_j = \langle \sigma_1, \dots, \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j \rangle.$$

If we use (7) and Lemma 3 repeatedly, it holds that

$$\tilde{s}_j = s_{j-1} + a_j \sum_{1 < i < j-1} |s_j^i| s_{i-1} - l_j s_j, \tag{10}$$

where $s_j^i = \langle \sigma_i, \dots, \sigma_j \rangle$. Using (10), (a), and what has been proved for the indices $1 \leqslant i < j$ it follows that

$$0 = \int x^k \tilde{Q}_n(x)\, d\tilde{s}_j(x), \quad 0 \leqslant k < \tilde{n}_j = n_{j-1} \Leftrightarrow 0 = \int x^k \tilde{Q}_n(x)\, ds_{j-1}(x),$$
$$0 \leqslant k < n_{j-1}.$$

Finally, consider an index $i > j$. On the second system, this index is associated to the measure

$$\begin{aligned}\tilde{s}_i &= \langle \sigma_1, \dots, \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j, \langle \sigma_{j+1}, \sigma_j \rangle, \dots, \sigma_i \rangle \\ &= \langle \sigma_1, \dots, \langle \sigma_{j-1}, \sigma_j \rangle, \tau_j, g \langle \sigma_{j+1}, \sigma_j \rangle \rangle,\end{aligned}$$

where $g(x) \equiv 1$ for $i = j+1$ and $g(x) = \int (x-t)^{-1}\, ds_i^{j+2}(t)$ for $j+2 \leqslant i \leqslant m$. Using (8) it holds that

$$\tilde{s}_i = -s_i + A_g s_j, \tag{11}$$

where $A_g \in \mathbb{C}$ is as in Lemma 2. From (11), (b), and the orthogonality relations just proved for the index j it follows that for $i > j$

$$0 = \int x^k \tilde{Q}_n(x)\, d\tilde{s}_i(x), \quad 0 \leqslant k < \tilde{n}_i = n_i \Leftrightarrow 0 = \int x^k \tilde{Q}_n(x)\, ds_i(x), \quad 0 \leqslant k < n_i$$

with which we conclude the proof. □

Theorem 1. *Let* $\mathbf{f}$ *be a Nikishin system and* $n = (n_1, \dots, n_m)$ *a multi-index such that there do not exist* $1 \leqslant i < j < k \leqslant m$ *such that* $n_i < n_j < n_k$. *Then,* Q_n *has exactly* $|n| = n_1 + \cdots + n_m$ *simple zeros in* F_1. *Therefore,* $\deg Q_n = |n|$ *and the multi-index n is normal.*

Proof. If the components of n are decreasing in value, the conditions of Lemma 1 are satisfied and we have nothing to prove. Therefore, suppose that n has a component $j \geqslant 2$ for which $n_{j-1} < n_j$. Among all such components we take one for which n_j is largest in value. According to the assumptions of the theorem for this index j conditions (a) and (b) are fulfilled. We can apply Lemma 4 thus obtaining a pair $(\mathbf{f}^j, n^j)$ equivalent to $(\mathbf{f}, n)$. Notice that $(\mathbf{f}^j, n^j)$ also satisfies the assumptions of the theorem; that is n^j does not have three components increasing in value. So we can repeat the process. Obviously, after a finite number of repetitions we arrive at a multi-index whose components are decreasing in value and according to Lemma 1 the corresponding Q_n has exactly $|n|$ simple zeros all lying on F_1. But this polynomial Q_n is exactly the same one corresponding to the initial system since it satisfies the same set of orthogonality relations. With this we conclude the proof. □

The previous result has applications to the study of the convergence of Hermite–Padé approximation to Niskishin systems of functions. We have:

Corollary 1. *Let* $\mathbf{f} = (f_1, \dots, f_m)$ *be a Nikishin system. Let* $\{n(r)\}, r \in \mathbb{N}$, *be a sequence of multi-indices* $n(r) = (n_1(r), \dots, n_m(r)), \lim_{r\to\infty} |n(r)| = \infty$, *such that for each* $r \in \mathbb{N}$ *there do not exist* $1 \leqslant i < j < k \leqslant m$ *such that* $n_i(r) < n_j(r) < n_k(r)$ *and there exists a constant* c *such that* $n_i(r) \geqslant (|n(r)|/m) - c$ *for all* $1 \leqslant i \leqslant m$. *Assume that either* F_2 *is bounded or* $\sum_{v=1}^{\infty} c_v^{-1/2v} = \infty$ *where* $c_v = \int |x|^v \, d\sigma_1(x)$. *Then for each* $1 \leqslant i \leqslant m$

$$\lim_{r\to\infty} \frac{P_{n(r),i}}{Q_{n(r)}} = f_i$$

uniformly on each compact subset of $\mathbb{C} \backslash F_1$.

Proof. According to [2, Theorem 1], our assumptions imply that each component of the Hermite–Padé approximant converges to the corresponding component of $\mathbf{f}$ in logarithmic capacity on each compact subset of $\mathbb{C} \backslash F_1$. On the other hand, for all $1 \leqslant i \leqslant m$ and $r \in \mathbb{N}$, all the poles of $P_{n(r),i}/Q_{n(r)}$ lie on F_1. According to [4, Lemma 1], this and the convergence in capacity imply our statement. □

Another consequence extends Corollary 2 in [2].

Corollary 2. *Let* $\mathbf{f} = (f_1, \dots, f_m)$ *be a Nikishin system. Let* $\{n(r)\}, r \in \mathbb{N}$, *be a sequence of multi-indices* $n(r) = (n_1(r), \dots, n_m(r)), \lim_{r\to\infty} |n(r)| = \infty$, *such that for each* $r \in \mathbb{N}$ *there do not exist* $1 \leqslant i < j < k \leqslant m$ *such that* $n_i(r) < n_j(r) < n_k(r)$ *and there exists* $j, 1 \leqslant j \leqslant m$, *such that for all* $r \in \mathbb{N}$ *we have that* $n_j(r) \geqslant n_i(r), i \neq j$. *Assume that either* F_2 *is bounded or* $\sum_{v=1}^{\infty} c_v^{-1/2v} = \infty$. *Then*

$$\lim_{r\to\infty} \frac{P_{n(r),j}}{Q_{n(r)}} = f_j$$

uniformly on each compact subset of $\mathbb{C} \backslash F_1$.

Proof. The proof is basically the same as for Corollary 1 except that the arguments can only be applied to the jth component. □

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